# HOSSAM GHANEM

## (25) 3.10 Linear Approximations And Differentials

$$\Delta y \approx f^{\backslash}(x) \cdot \Delta x$$

$$f(x + \Delta x) \approx f(x) + f^{\backslash}(x) \cdot \Delta x$$

**Example 1**
31 July 31st , 2003

Let $f(x) = \sqrt{x}$
(a) Find $dy$ at $x = 9$ with $\Delta x = -0.1$
(b) Use differentials to approximate $f(8.9)$

### Solution

$f(x) = \sqrt{x} \quad \rightarrow \quad f(9) = 3$

$f^{\backslash}(x) = \dfrac{1}{2\sqrt{x}} \quad \rightarrow \quad f^{\backslash}(9) = \dfrac{1}{6} = 0.167$

$dy \approx f^{\backslash}(x) \cdot \Delta x$

$dy \approx 0.167\,(-0.1) = -0.0167$

$f(x + \Delta x) \approx f(x) + f^{\backslash}(x) \cdot \Delta x$

$f(8.9) \approx f^{\backslash}\big(9 + (-0.1)\big) \approx f(9) + f^{\backslash}(9)(-0.1) \approx 3 + (-0.0167) \approx 2.9833$

**Example 2**
10 June 6 1994

Use differentials to find an approximate value for: $\sqrt[3]{26.9}$

### Solution

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \quad \rightarrow \quad f(27) = 3$

$f^{\backslash}(x) = \dfrac{1}{3} x^{\frac{-2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}} \quad \rightarrow \quad f^{\backslash}(27) = \dfrac{1}{3(9)} = \dfrac{1}{27} = 0.037$

$\Delta x = 26.9 - 27 = -0.1 = \dfrac{-1}{10}$

$f(x + \Delta x) \approx f(x) + f^{\backslash}(x) \cdot \Delta x$

$f(26.9) \approx f\big(27 + (-0.1)\big) \approx f(27) + f^{\backslash}(27)(-0.1) \approx 3 + \left(\dfrac{1}{27} \cdot \dfrac{-1}{10}\right) \approx 3 - \dfrac{1}{270}$

| Example 3<br>8 August 28 , 1993 | Use differentials to find an approximate value for:<br>$\sqrt{(1.04)^3+3} \; + \; (1.04)^{\frac{2}{3}}$ |
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## Solution

Let $f(x) = \sqrt{x^3+3} + x^{\frac{2}{3}}$ $\rightarrow$ $f(1) = \sqrt{4} + 1 = 3$

$f^{\backslash}(x) = \dfrac{3x^2}{2\sqrt{x^3+3}} + \dfrac{2}{3}x^{\frac{-1}{3}}$ $\rightarrow$ $f^{\backslash}(1) = \dfrac{3}{2\sqrt{4}} + \dfrac{2}{3} = \dfrac{3}{4} + \dfrac{2}{3} = \dfrac{9+8}{12} = \dfrac{17}{12}$

$\Delta x = 1.04 - 1 = 0.04 = \dfrac{4}{100} = \dfrac{1}{25}$

$f(x+\Delta x) \approx f(x) + f^{\backslash}(x)\cdot \Delta x$

$f(1+0.04) \approx f(1) + f^{\backslash}(1)\cdot \Delta x$

$f(1.04) \approx 3 + \dfrac{17}{12}\cdot\dfrac{1}{25}$

$\sqrt{(1.04)^3+3} + \left((1.04)^{\frac{2}{3}}\right) \approx 3 + \dfrac{17}{300} \approx 3\dfrac{17}{300}$

| Example 4<br>47 December ,22 2009 | Use linear approximation to estimate the number<br>$N = \dfrac{\sqrt{2-(1.001)^2}}{1.001}$ |
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## Solution

Let $f(x) = \dfrac{\sqrt{2-x^2}}{x}$ $\qquad f(1) = \dfrac{\sqrt{2-1}}{1} = 1$

$f^{\backslash}(x) = \dfrac{x\cdot\dfrac{-2x}{2\sqrt{2-x^2}} - \sqrt{2-x^2}}{x^2}$ $\qquad f^{\backslash}(1) = \dfrac{\dfrac{-2}{2} - 1}{1} = -2$

$\Delta x = 1.001 - 1 = 0.001 = \dfrac{1}{1000}$

$f(x+\Delta x) \approx f(x) + f^{\backslash}(x)\cdot \Delta x$

$f(1+0.001) \approx f(1) + f^{\backslash}(1)\cdot \Delta x$

$f(1.001) \approx 1 + (-2)\left(\dfrac{1}{1000}\right)$

$\dfrac{\sqrt{2-(1.001)^2}}{1.001} \approx 1 - \dfrac{2}{1000} = 0.998$

## Example 5
35 December 16, 2004

Let $f(x) = \dfrac{\sqrt{3+x}}{x}$

Use differentials to find an approximate value for $\dfrac{\sqrt{3.9}}{0.9}$

### Solution

$$f(x) = \frac{\sqrt{3+x}}{x} \qquad f(1) = \frac{\sqrt{3+1}}{1} = 2$$

$$f'(x) = \frac{x \cdot \frac{1}{2\sqrt{3+x}} - \sqrt{3+x}}{x^2} \qquad f'(1) = \frac{\frac{1}{2\sqrt{4}} - \sqrt{4}}{1} = \frac{1}{4} - 2 = -\frac{7}{4}$$

$$\Delta x = 0.9 - 1 = -0.1 = \frac{-1}{10}$$

$$f(x+\Delta x) \approx f(x) + f'(x)\cdot\Delta x$$

$$f\big(1+(-0.1)\big) \approx f(1) + f'(1)\cdot\Delta x$$

$$\frac{\sqrt{3+0.9}}{0.9} \approx 2 + \left(\frac{-7}{4}\right)\left(\frac{-1}{10}\right) = 2 + \frac{7}{40}$$

$$\frac{\sqrt{3+0.9}}{0.9} = 2\frac{7}{40}$$

## Example 6
29 July 25th, 2002

Let $y = \dfrac{\sqrt{x}}{x-2}$ Use differentials to approximate the change in $y$ if $x$ changes from 4 to 3.9

### Solution

$$f(x) = \frac{\sqrt{x}}{x-2} \qquad f(4) = \frac{\sqrt{4}}{4-2} = 1$$

$$f'(x) = \frac{(x-2)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x-2)^2} \qquad f'(4) = \frac{(4-2)\cdot\frac{1}{2\sqrt{4}} - \sqrt{4}}{(4-2)^2} = \frac{\frac{1}{2} - 2}{4} = \frac{3/2}{4} = \frac{3}{8}$$

$$\Delta x = 3.9 - 4 = -0.1 = \frac{-1}{10}$$

$$\Delta y \approx f'(x)\cdot\Delta x \approx \frac{3}{8}\cdot\left(\frac{-1}{10}\right) = \frac{-3}{80}$$

## the linear approximation

$$f(x) = f(a) + f^{\backslash}(a)(x-a)$$

| Example 7<br>45 May 10, 2009 | Find the linear approximation of $f(x) = \sqrt{2x+7}$ at $a = 1$ . |
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## Solution

$f(x) = \sqrt{2x+7}$

$f^{\backslash}(x) = \dfrac{2}{2\sqrt{2x+7}} = \dfrac{1}{\sqrt{2x+7}}$

$f(1) = \sqrt{2(1)+7} = \sqrt{9} = 3$

$f^{\backslash}(1) = \dfrac{1}{\sqrt{2(1)+7}} = \dfrac{1}{3}$

$f(x) = f(a) + f^{\backslash}(a)(x-a)$

$f(x) = 3 + \frac{1}{3}(x-1)$

| Example 8<br>52 July 23 , 2011 A | [ 2 + 1 Points ] Let $f(x) = 1 + \sin x$.<br>a) Use differentials to find expressions for $\Delta y$ and $dy$<br>b) Use the linear approximation formula to estimate $f(29°)$ |
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## Solution

a)

$f(x) = 1 + \sin x$

$f(x+\Delta x) = 1 + \sin(x+\Delta x)$

$\Delta y = f(x+\Delta x) - f(x) = 1 + \sin(x+\Delta x) - (1+\sin x) = \sin(x+\Delta x) - \sin x$

$dy = f'(x)\,dx = \cos x\,dx$

b)

$f(x) = 1 + \sin x$ $\qquad f\left(\frac{\pi}{6}\right) = 1 + \sin\frac{\pi}{6} = 1 + \frac{1}{2} = \frac{3}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$f(x) = f(a) + f^{\backslash}(a)(x-a)$

$$f\left(\frac{29\pi}{180}\right) = f\left(\frac{\pi}{6}\right) + f^{\backslash}\left(\frac{\pi}{6}\right)\left(\frac{29\pi}{180} - \frac{\pi}{6}\right) = \frac{3}{2} + \frac{\sqrt{3}}{2}\left(\frac{-\pi}{180}\right) = \frac{3}{2} - \frac{\sqrt{3}\,\pi}{360}$$

# Homework

| | | |
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| | Use differentials to find an approximate value for | |
| 1 | $\sqrt{3.91}$ | 2 June 12 1990 |
| 2 | $(-26.88)^{\frac{2}{3}}$ | 1 December 3, 1992 |
| 3 | $1+(7.9)^{\frac{2}{3}}$ | 2 May 20, 1993 |
| 4 | $(1.001)^2+(1.001)^{\frac{4}{3}}+4$ | |
| 5 | $\sqrt{(3.02)^3-2}$ | 6 January 6, 1993 |
| 6 | $\sqrt{7+\sqrt[3]{7.9}}$ | 11 August 11, 1994 A |
| 7 | $(9.03)^5+\sqrt{9.03}$ | 1 November 1987 |
| 8 | $\sqrt[3]{28}$ | 46 August 1 , 2009 |
| 9 | $\sqrt[3]{1.03}$ | 44 December 21, 2008 |
| 10 | $\sqrt[4]{15.9}$ | 36 Dec 15, 2005 |
| 11 | $\sqrt[3]{8.12}$ | 42 May 5, 2008 |
| 12 | $\frac{1}{(0.98)^{10}}$ | 40 May 3, 2007 |
| 13 | $7+(1.02)^4$ | 41 July 19, 2007 |
| 14 | $1-\frac{1}{\sqrt[3]{8.01}}$ | 39 December 14, 2006 |
| 15 | $1+(8.01)^{\frac{2}{3}}$ | 37 May 4, 2006 |

# Homework

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| 16 | $\sqrt{1.1}+\sqrt[3]{1.1}$ | 34 July 22, 2004 |
| 17 | $\sqrt{(1.01)^2+3}$ | 28 Dec 20, 2001 |
| 18 | $(8.01)^{\frac{4}{3}}+(8.01)^2$ | |
| 19 | $\sqrt{(2.02)^3+1}$ | 26 May 10, 2001 |
| 20 | 43 July 19, 2008<br>Use differentials to find an approximate value for: $\sec(46)^{\circ}$ | |
| 21 | 38 July 17, 2006<br>Use differentials to find an approximate value for: $\sin(29)^{\circ}$ | |
| 22 | 33 May 6, 2004<br>Let $f(x)=3(4+x)^{\frac{2}{3}}$ Use differentials to approximate value $f(3.9)$ | |
| 23 | 30 May 15th, 2003<br>Let $f(x)=\sqrt{x-3}$ Use differentials to approximate $f(11.7)$ | |
| 24 | 31 July 31st , 2003<br>Let $f(x)=\sqrt{x}$.<br>a) Find $dy$ at $x=9$ with $\Delta x=-0.1$.<br>b) Use differentials to approximate $f(8.9)$. | |
| 25 | 20 January 3 ,2001<br>Let $f(x)=x^3+\frac{1}{\pi}\sec^2\left(\frac{\pi}{8}x\right)$ . Use differentials to find the approximate change in $f$ if $x$ changes from 2 to 2.01 (3 points) | |
| 26 | 18 May 24 ,2000<br>Use differentials to approximate the change in $y=\frac{x}{\sqrt[3]{x^2+2}}$, when $x$ changes from 5 to 4.9. | |
| 27 | 27 August 2, 2001<br>Use linear approximation to estimate $f(7.1)$ where $f(x)=3(1+x)^{\frac{1}{3}}$ (3 pts.) | |

# Homework

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|---|---|---|
| 28 | Let $$f(x) = \sin^2(x^2 - 3x + \pi/4)$$ (a) Find $f^{\backslash}(x)$<br>(b) Use linear approximation to estimate $f(0.02)$. | 48 Sunday 9 May 2010 |
| 29 | (3 Points) Find the value of $dy$ if $$y = x^2 + \frac{1}{\pi}\sec^2\left(\frac{\pi}{12}x\right)$$ and $x$ changes from 3 to 3.03 | 49 July 24, 2010 |
| 30 | (3 pts.) Use differentials to approximate $4 - (8.1)^{1/3}$ | 50 22 December 2010 |
| 31 | [3 pts.] Use linear approximation to estimate $\sqrt{16.08}$. | 51 8 May 2011 |
| 32 | Use differentials to find an approximate value for: $\sec^2(44.9^\circ)$ | 32 December 18, 2003 |
| 33 | Let $y = \tan x$<br>(a) Use differentials to find $dy$ if $x$ changes from $45^\circ$ to $44^\circ$<br>(b) Use linear approximation to estimate $\tan 44^\circ$ | 24 July 20th , 2000 |
| 34 | (4 points ) : Use linear approximation to estimate the number $(0.94)^{1/3}$ | 07/12/2011 |

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| 32<br>32 December 18, 2003 | Use differentials to find an approximate value for: $\sec^2(44.9^\circ)$ |

## Solution

$f(45^\circ) =$

Let

$f(x) = \sec^2 x$

$f\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$

$f^\backslash(x) = 2\sec x \ \sec x \ \tan x = 2\sec^2 x \ \tan x$

$f^\backslash\left(\frac{\pi}{4}\right) = 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi^\circ}{4}\right) = 2.\left(\sqrt{2}\right)^2.1 = 4$

$\Delta x = \frac{(44.9-45)\pi}{180} = \frac{-0.1\pi}{180} = \frac{-\pi}{1800}$

$f(x+\Delta x) \approx f(x) + f^\backslash(x)\cdot\Delta x$

$f\left(\frac{\pi}{4}+\left(\frac{-\pi}{1800}\right)\right) \approx f\left(\frac{\pi}{4}\right) + f^\backslash\left(\frac{\pi}{4}\right)\cdot\Delta x = 2 + 4\left(\frac{-\pi}{1800}\right) = 2 - \frac{\pi}{450}$

$\sec^2\left(\frac{\pi}{4}+\left(\frac{-\pi}{1800}\right)\right) = \frac{900}{450} - \frac{\pi}{450}$

$\sec^2(44.9^\circ) \approx \frac{900-\pi}{450}$

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| 33<br>24 July 20th , 2000 | Let $y = \tan x$<br>(a) Use differentials to find $dy$ if $x$ changes from $45^\circ$ to $44^\circ$<br>(b) Use linear approximation to estimate $\tan 44^\circ$ |

## Solution

$f(x) = \tan x$

$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f^\backslash(x) = \sec^2 x$

$f^\backslash\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2$

$\Delta x = \frac{(44-45)\pi}{180} = \frac{-\pi}{180}$

$dy \approx f^\backslash(x)\cdot\Delta x$

$\approx 2\cdot\frac{-\pi}{180} = \frac{-\pi}{90}$

$f(x) = f(a) + f^\backslash(a)(x-a)$

$f\left(\frac{44\pi}{180}\right) = f\left(\frac{\pi}{4}\right) + f^\backslash\left(\frac{\pi}{4}\right)\left(\frac{44\pi}{180}-\frac{\pi}{4}\right)$

$\tan 44^\circ = 1 + 2\left(\frac{44\pi}{180}-\frac{45\pi}{180}\right) = 1 - \frac{\pi}{90}$